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BIOGRAPHY.

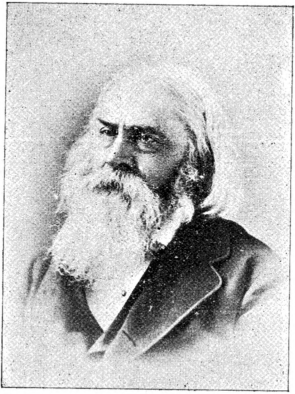
BENJAMIN PEIRCE.

By F. P. MATZ, M Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

BENJAMIN PEIRCE was born at Salem, Massachusetts, April 4, 1809, and died at Cambridge, Massachusetts, October 6, 1880. He entered Harvard College, at the age of sixteen; and, at the age of twenty, he was graduated from the same College, with highest honors. He devoted himself principally to the study of Mathematics. This *favorite* study of his was pursued far beyond the limits of the curriculum of mathematical studies prescribed by the authorities of Harvard College, at that time.

As an under-graduate student, young Peirce was instructed by Nathaniel Bowditch, who soon perceived the innate mathematical genius of his pupil. Bowditch proudly predicted the future greatness of the young man. Not only did Bowditch give him valuable instruction in geometry and analytics, but also acted as his *mathematical adviser*—carefully directing him in the development of his mathematical talents and scientific powers. The lectures on higher mathematics delivered by Francis Grund he was enabled to attend, by reason of his preparation beyond the limit of the under-graduate course in mathematics. When Dr. Bowditch was publishing his translation and commentary of the *Mechanique Celeste* of Laplace, young Peirce assisted in reading the proof-sheets. This critical reading of that great work of Laplace was to him an education in itself, and may have been the prime cause that not a small part of Peirce's subsequent mathematical and scientific work was done in the great field of analytical mechanics:

In the class-room, he frequently gave original demonstrations which



Benjamin Peirce.

proved to be more direct and scientific than those given in the text-books of that day. On graduating, he went to Northampton, Massachusetts, as a teacher in Mr. Bancroft's School. As *tutor*, he returned to Harvard College, in 1831. Since Professor Farrar spent the next year in Europe, tutor Peirce was left at the *head* of the Department of Mathematics in Harvard College; and, on account of the physical inability of Professor Farrar to resume teaching, Peirce *continued* to fill his place. In fact, Peirce held this position, advancing step by step, until the time of his death. His position, in 1842, was christened "The Perkins Professorship of Mathematics and Astronomy." In the history of mathematical teaching at Harvard College, the year 1833 marks an important epoch; as it was then that Benjamin Peirce became the *professor* of Mathematics and Natural Philosophy in that institution of learning.

Professor Peirce was married in July, 1833. At the time of his death, there were living his wife, three sons, and a daughter. His eldest son, James M. Peirce, is University professor of mathematics in Harvard; Charles S Peirce is a professor in the Johns Hopkins University; and H. H. D. Peirce is connected with the firm of Herter Brothers, New York City.

It has been said that a mere boy detected an error in Bowditch's solution of a problem. "Bring me the boy who corrects my mathematics, said Bowditch. Master Benjamin Peirce was the boy who had done the correcting; and thirty years later, this same Benjamin Peirce dedicated one of his great mathematical works "To the cherished and revered memory of my master in science, Nathaniel Bowditch, *The Father of American Geometry*." This same title was bestowed upon Peirce, by foreign mathematicians. Sir Wm. Thomson (Lord Kelvin), in an address before the British Association, referred to Benjamin Peirce as "*The Founder of High Mathematics in America*"; and on a similar occasion, the late Professor Cayley referred to him as "*The Father of American Mathematics*." The name of Benjamin Peirce is that of an *American mathematician*, whom no one need hesitate to rank with the names of Pythagoras, Leibnitz, Newton, Legendre, John Bernoulli, Wallis, Abel, Laplace, Lagrange, and Euler. Through the united efforts of the late Professor Wm. Chauvenet (Yale's ablest mathematician and astronomer) and Benjamin Peirce —not to speak of their worthy successors, was effected the general adoption of the *ratio-system* in American works on trigonometry.

In the reforms incident to the *New Education*, Harvard has always taken a prominent part and Benjamin Peirce was *an enthusiastic advocate* of the elective system with respect to collegiate studies. As a branch of Harvard College, there was opened, in 1842, the Lawrence Scientific School; and in this school, Professor Peirce gave instruction in higher Mathematics including analytical and celestial mechanics. Such advanced courses of mathematics, as *he* offered to students, in 1848, had never before been offered to American students by any other professor in any other American college. The second American educational institution which offered equally advanced courses of mathematics, is the Johns Hopkins University; and these courses were arranged by that *English master*, who gave a fresh and powerful impulse to mathe-

matical study and teaching in America—*Professor J. J. Sylvester.*

The preparation of mathematical text-books was begun by Professor Peirce, immediately on beginning his career as teacher of Mathematics in Harvard College. In 1835 appeared his *Elementary Treatise on Plane Trigonometry*; in 1836, his *Elementary Treatise on Spherical Trigonometry* together with his *Elementary Treatise on Sound*; in 1837, his *Elementary Treatise on Plane and Solid Geometry* together with his *Elementary Treatise on Algebra*; during the period 1841-46, he wrote and published in *two* volumes his *Elementary Treatise on Curves, Functions, and Forces;* and in 1855, he published his *Analytical Mechanics*. Subsequently was published his memoir on *Linear Associative Algebra*; and this memoir, according to Professor James Mills Peirce, he regarded as his great work. All of his works are models of conciseness, perspicuity, and elegance; and they all evince extraordinary originality and genius.

In 1867, Professor Peirce was made the Superintendent of the United States Coast Survey; and he held that position for seven years. He had been consulting astronomer to the *American Ephemeris and Nautical Almanac*, since 1849; and for many years, he *directed* the theoretical part of the work. In 1855, Professor Peirce was one of the men intrusted with the organization of the Dudley Observatory. For many years *before* and *after* he took charge of the United States Coast Survey, he was frequently consulted with respect to the work in that office. He received the degree of *Doctor of Laws* from the University of North Carolina, in 1847, and also from Harvard University in 1867. He was elected an Associate of the Royal Astronomical Society of London in 1849 and a member of the Royal Society of London in 1852. He was elected president of the American Association for the Advancement of Science, in 1853 (the fifth year of its existence); and he was one of the *original* members of the Royal Societies of Edinburg, and Gottingen; Honorary Fellow of the Imperial University of St. Vladimir, at Kiev;etc.

Professor Peirce's conception of the American Social Science Association was that it should be a *university for the people*,—combining those who can contribute any thing original in social science into a temporary academical senate, to meet for some weeks in a given place and debate questions with each other, as well as to give out information for the public. In this line of thought he favored, also,the establishment of the Concord School of Philosophy, to do a similar work in the speculative studies; and he lived to see the partial realization of what he foresaw in this instance. In a Mathematical Society over which he presided for some years, each member would bring something novel in his own particular branch of study; and in the discussion which followed, it would almost invariably appear that Professor Peirce had, while the paper was being read, pushed out the author's methods to far wider results than the author had dreamed possible. The same power of extending rapidly in his own mind novel mathematical researches was exhibited at the sessions of every scientific body at which he chanced to be present. What was quite as admirable was the way in which he did it, giving

the credit of the thought always to the author of the essay under discussion. His pupils thus frequently received credit for what was in reality far beyond their attainment. He robbed himself of fame in two ways: by giving the credit of his discoveries to those who had merely suggested the line of thought, and by neglecting to write out and publish that which he had himself thought out.

In physical astronomy, perhaps, his greatest works were in connection with the planetary theory, his analysis of the Saturnian system, his researches regarding the lunar theory, and the *profound criticism* of the discovery of Neptune following the investigations of Adams and Leverrier. At the time of the publication of his "*System of Analytical Mechanics*," Professor Peirce announced that the volume would be followed by three others, entitled respectively: "*Celestial Mechanics*," "*Potential Physics*," and "*Analytical Morphology*." These three volumes were never published.

Professor Peirce, in a paper read before the American Association for the Advancement of Science, in 1849, showed in the vegetable world the demonstrable presence of an intellectual plan—showed that phyllotaxis (the science of the relative position of leaves) involved an algebraic idea; and this algebraic ida was subsequently shown to be the solution of a physical problem.

The higher mathematical labors of so eminent a geometer must lie beyond the course of general recognition. Among the things which give him a just claim to this title, may be mentioned: his discussion of the motions of two pendulums attached to a horizontal cord; of the motions of a top; of the fluidity and tides of Saturn's rings; of the forms of fluids enclosed in extensible sacs; of the motions of a sling; of the orbits of Uranus, Neptune, and the comet of 1843; of the criteria for rejecting doubtful observations; of a new form of binary arithmetic, of *systems* of linear and associative algebra; of various mechanical games, puzzles, etc.; of various problems in geodesy; of the lunar tables; of the occultations of the Pleiades; etc. He adapted the epicycles of Hipparchus to the analytical forms of modern science; and he, also, solved by a system of co-ordinates of his own devising, several problems concerning the involutes and evolutes of curves, which would probably have proved impregnable by any other method of mathematical approach.

None of Professor Peirce's labors lie farther above the ordinary reach of thought than his little lithographed volume on Linear and Associative Algebra. In this he discusses the nature of mathematical methods, and the characteristics which are necessary to give novelty and unity to a calculus. Then he passes to a description of seventy or eighty different kinds of simple calculus. Almost no comment is given; but the mathematical reader discovers, as he proceeds, that only *three* species of calculus, having each a unity in itself, have been hitherto used to any great extent,—namely, *ordinary algebra*, *differentials*, and *quaternions*. Think of it; what a wonderful volume of prophecy that is which describes seventy or eighty species of algebra, any one of which would require generation after generation of ordinary mathematicians to develop!

On both sides of the Atlantic, Professor Peirce as an author, was highly esteemed. His work on analytical mechanics was, at the time of its publication, regarded even in Germany, as the *best* of its kind. As a lecturer, Professor Peirce was highly esteemed in both scientific and popular circles. It is related that in 1843, by a series of popular lectures on astronomy, he so excited the public interest that the necessary funds were immediately supplied, for erecting an astronomical observatory at Harvard College. A remarkable series of lectures on "*Ideality in Science*," delivered by him in 1879 before the Lowell Institute in Boston, attracted the general attention of American thinkers, on account of the thoughtful consideration of the vexed question of science and religion.

Professor Peirce was a transcendentalist in mathematics, as Agassiz was in zoology; and a certain subtile tie of affinity connected these two great men, however unlike they were in their special genius. Alike, also, they were in their enthusiasm which neither the piercing scepticism of Cambridge could wither, nor declining years chill with the frost of age. The thing he distrusted was routine and fanatical method, whether new or old; for thought, salient, vital, co-operative thought, in novel or in ancient aspects, he had nothing but respect and furtherance. Few men could suggest more while saying so little, or stimulate so much while communicating next to nothing that was tangible and comprehensible. The young man who would learn the true meaning of *apprehension* as distinct from *comprehension*, should have heard the professor lecture, after reciting to him. He was always willing to be esteemed for less than he had really accomplished; and he could join most heartily in the praise of others who even owed their impulse to him. *Modest* and *magnanimous*, but not unobservant, his ambition for personal distinction was early and easily satisfied; and he thus rid himself of what is to most men a perturbing, and too often an ignoble, element of discomfort.

Professor Peirce habitually ascribed to his listener a power of assimilation which the listener rarely possessed. He assumed his readers could follow wherever he led; and this made his lectures hard to follow, his books brief, difficult, and comprehensive. When, however, his listeners were students who had previously attained some skill as mathematicians and who had been trained in his own methods, the resulting work would be of *the highest order of excellence.* He was personally magnetic in his presence. His pupils loved and revered him; and to the young man, he always lent a helping hand in science. He inspired in them a love of truth for its own sake.

His own faith in Christianity had the simplicity of a child's; and whatever radiance could emanate from a character which combined the greatest intellectual attainment with the highest moral worth, that radiance cast its light upon those who were in his presence. "*Every portion of the material universe*," writes Professor Peirce, "*is pervaded by the same laws of mechanical action which are incorporated into the very constitution of the human mind.*" To him, then, the universe was made for the instruction of man. With this belief he approached the study of natural phenomena not in the spirit of a critic, but

reverently in the mood of a sympathizing reader; and the lesson he reads is: "*There is but one God, and science is the knowledge of Him.*" In his lectures and teaching he showed, as he always felt with adoring awe, that the mathematician enters (as none else can) into the intimate thought of God, sees things precisely as they are seen by the Infinite Mind, holds the scales and compasses with which the Eternal Wisdom built the earth and meted out the heavens. This consciousness had pervaded his whole scientific life. It was active in his early youth, as his coevals well remember; it gathered strength with his years; and it struck the ever recurring key-note in his latest public utterances.

Benjamin Peirce was a devout, God-fearing man; he was a Christian, in the whole aim, tenor, and habit of his life. To know Professor Peirce was simply to love him, to admire him, and to revere him. Since he was conversant with the phases of scientific infidelity, and by no means unfamiliar with the historic grounds of scepticism, it can not be regarded otherwise than with the profoundest significance, that a *mind* second to none in keen intuition, in aesthetic sensibility, in imaginative ferver, and in the capacity of close and cogent reasoning, *maintained* through life an unshaken belief and trust in the power, providence, and love of God, as beheld in his works, and as incarnate in our Lord and Savior. In one of his lectures on *Ideality in Science*, he said: "Judge the tree by its fruit." Is this magnificent display of ideality a human delusion? Or is it a divine record? The heavens and the earth have spoken to declare the glory of God. It is not a tale told by an idiot, signifying nothing. It is the poem of an infinite imagination, signifying immortality."

In May, 1880, Professor Peirce began to pass under the shadow of the cloud of his last illness. For some weeks there was little serious fear that it was a shadow not destined to lift. He was first confined to his chamber, on the 25th of June, 1880; and from that time, his slowly failing condition was hardly relieved even by any deceptive appearances of improvement. He died on the morning of Wednesday, October 6, 1880. Distinguished throughout his life by his freedom from the usual abhorrence of death, which he never permitted himself either to mourn when it came to others, or to dread for himself, he kept this characteristic temper to the end, through all the sad changes of his trying illness; and, two days before he ceased to breathe, it struggled into utterance in a few faintly-whispered words, which expressed and earnestly inculcated a cheerful and complete acceptance of the will of God with regard to him.

The funeral took place on Saturday, October 9, 1880, at Appleton Chapel, and was the occasion of an impressive gathering of people of great and various mark. The attendance included a very full representation of the various faculties and governing boards of the University; a large deputation of officers of the United States Coast and Geodetic Survey, headed by the superintendent and the chief assistant; delegations of eminent professors from Yale College and the Johns Hopkins University; many members of the class of 1829; and a great number of other friends of the deceased.

The pall-bearers were: President Charles W. Eliot; Ex-President Thomas Hill, Pastor of the First Parish Church, Portland, Maine; Capt. C. P. Patterson, Superintendent of the United States Coast Survey; Professor J. J. Sylvester, of the Johns Hopkins University; Hon. J. Ingersoll Bowditch; Professor Simon Newcomb, Superintendent of the American *Ephemeris and Nautical Almanac*; Dr. Oliver Wendell Holmes; Professor Joseph Lovering; and Dr. Morrill Wyman. A beautiful and simple service was conducted by the Rev. A. P. Peabody and the Rev. James Freeman Clarke.

In the career of Professor Benjamin Peirce, America has nothing to regret, but that it is now closed; while the American people have much to learn from his long, useful, and honorable life.

REMARKS ON SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Professor of Mathematics, University of Michigan, Ann Arbor, Michigan.

[Continued from May Number.]

Since $a.bbc=acb$ and bc, $ab=abc$ it follows that the result is not always independent of the order in which we perform the operations indicated by two substitutions. In the equation $ab.bc=acb$ we call ab and bc the *factors* and acb the *product* and the process is called *multiplication of substitutions*. The above example shows that the law that the product is independent of the order of the factors does not hold true with respect to the multiplication of substitutions.

The number of substitution groups increases very rapidly as the number of letters increases. During the last few years the work of making complete lists of such groups has been carried through ten letters* but no formula has yet been published by means of which the number of such groups can readily be determined for any number of letters.

If an expression involving a given number of letters is unchanged by applying all the substitutions of a group of the same number of letters to it but is changed by applying any other substitution of the same or a lower number

*The number of groups of ten letters exceed one thousand. Complete lists are found in the Quartly Journal of Mathematics as follows: Cayley: Substitution groups for two, three, four, five, six, seven, and eight letters, vol. 25, pp. 71-88, 137-155. Cole: List of substitution groups of nine letters, vol. 26, pp. 372-388. Cole: List of transitive substitution groups of ten and eleven letters, vol. 27, pp. 39-50. Miller: Intransitive groups of ten letters, vol. 27, pp 99-118.

A few errors and omissions with respect to these lists have been noted in late numbers of the Bulletin of the American Mathematical Society. The lists are complete in the sense that an effort is made to give all the possible groups of the given number of letters and fairly accurate results have been attained.

of letters, then the expression is said to belong to the given group. We may take, for example, the expression with four letters

$$ac+bd.$$

We see that no matter what values a, b, c, and d may have this expression cannot change its value for any of the substitutions in

$$\begin{array}{llll} 1 & ab.cd & abcd & ac \\ & ac.bd & adcb & bd \\ & ad.bc & & \end{array}$$

but that it changes its value, in general, for any other substitution of four or a lower number of letters; hence we say the given expression belongs to this group, and, conversely, that the group belongs to this expression. From this it can be seen that substitution groups furnish a means by which we may classify such algebraic expressions and thus study the common properties once for all. It has been proved that every integral expression belongs to some group and that an infinite number of such expressions belong to each group. By studying a group we therefore study some properties common to an infinite number of algebraic expressions and from this it follows that the study of substitution groups is a matter of economy in case familiarity with a large number of expressions is to be attained.

The groups of two and three letters are so simple that they are frequently employed without any explanation of their connection with an extensive science. For example, when a factor of an expression belonging to one of these groups can be found by inspection and belongs to the same group as the expression the form of the other factor is often obtained from the fact that it must belong to the same group. This is explained by employing simple properties of these groups in place of the groups themselves. Even in these cases a knowledge of the theory of substitution groups would contribute much to the clear understanding of the matter on the part of the student and in the more complex cases such a knowledge becomes almost indispensible if a comprehensive knowledge is to be attained. In the factoring known as the solution of equations, substitution groups have since the time of Galois played the most prominent part—serving not only to give a comprehensive view of the entire field but also to extent the knowledge with respect to it.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED. A. M., (Princeton), Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the M.y Number.]

PROPOSITION XIX. *Let there be any triangle AHD (fig.18) right angled at H. Then in AD produced the portion DC is assumed equal to this AD; and the perpendicular CB is let fall to AH produced. I say hence will be established the hypothesis of right angle, or obtuse angle, or acute angle, according as the portion HB is equal to, or greater, or less than this AH.*

PROOF. For the join DB will be (Eu.I.4, and P. X of this) either equal to, or greater, or less than AD, or DC, according as the portion HB is equal to, or greater, or less than AH. And first indeed let HB be equal to AH, so that therefore the join DB may be equal to AD, or DC.

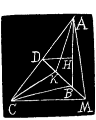


Fig. 18.

It follows that the circumference of the circle, which is described with the center D, and radius DB, will go through the points A, and C. Therefore the angle ABC, which is assumed right, is in this semicircle, whose diameter is AC. Wherefore (from the preceding proposition) is established the hypothesis of right angle. Quod erat primo loco demonstrandum.

Secondly let BH be greater than AH, so that therefore the join DB is greater than AD, or DC. It follows that the circumference of the circle, which is described with center D, and radius DA, or DC, will meet DB in some intermediate point K. Therefore, AK, and CK being joined, the angle AKC will be obtuse, because greater (Eu. I. 21) than the angle ABC, which is assumed right. Wherefore (from the preceding proposition) is established the hypothesis of obtuse angle. Quod erat secundo loco demonstrandum.

Thirdly let BH be less than AH, so that therefore the join DB is less than AD, or DC. It follows that the circumference of the circle, which is described with center D, and radius DA, or DC, will meet in some point M this DB produced outwardly. Therefore AM, and CM being joined, the angle AMC will be acute, because less (Eu. I. 21) than the angle ABC, which is assumed right.

Therefore (from the preceding proposition) is established the hypothesis of acute angle. Quod erat tertio loco demonstrandum. Itaque constant omnia proposito.

[To be continued.]

THE REALIZATION OF IMAGINARY POINTS.

By WARREN HOLDEN, Professor of Mathematics, Girard College, Philadelphia, Pennsylvania.

In Salmon's Conic Sections, Art. 82, the author says of an imaginary point: "It is a purely analytical conception, which we do not attempt to represent geometrically," * * * * "but attention to these imaginary points is necessary," * * * * "we shall meet with many cases in which the line joining two imaginary points is real."

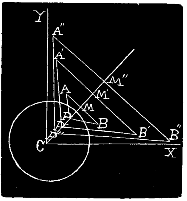
It is here proposed to trace some of these imaginary points, in the hope of finding them upon the real line which joins them.

Take the points of contact of tangents to a circle from a given point. The tangents are said to be real when the point from which they are drawn is without the circle, coincident when the point is on the circle, and imaginary when the point is within the circle. It is these last tangents and their points of contact, which it is proposed to find. These last points will be found, if any where, upon the polar of the given point, which is the "real line joining the imaginary points."

The co-ordinate of contact of tangent to a circle from any point $x'y'$, are $x''=\dfrac{R^2x'\pm Ry'\sqrt{x'^2+y'^2-R^2}}{x'^2+y'^2}$, $y''=\dfrac{R^2y'\mp Rx'\sqrt{x'^2+y'^2-R^2}}{x'^2+y'^2}$.

It will be shown that these formulas may be so interpreted as to determine the points of contact of tangents, from a point *within* the circle, to a curve which may be regarded as a mere development of the circle, and that the line joining these points is the polar of the given point.

When $x'y'$ reaches the circumference the expression under the radical sign becomes $x'^2+y'^2-R^2=0$ or $x'^2+y'^2=R^2$, which may be written $R^2-(x'^2+y'^2)$. *Proceeding now with the application of the formulas we obtain the following results:



Let C be the centre of a circle, CX, CY the axes, the radius equal to 4, and the distances of the points P, P', P'' from the centre respectively 3, 2, 1. Then for P or (x^1,y^1) we have

$$x''=\frac{16\times2.12\pm4\times2.12\sqrt{16-9}}{9}$$

$=6.25$ or 1.28: $y''=1.28$ or 6.25. These determine the points A, B. The line joining these points is the polar of P, as proved by $CM=\dfrac{R^2}{CP}=\dfrac{16}{3}=5\frac{1}{3}$. For P' we have $x''=\dfrac{16\times1.4\pm4\times1.41\sqrt{16-4}}{4}$

*By substituting the sign difference in place of the sign minus in the expression under the radical, the same result is reached.

$=10.44$ or 0.75, which give the points A', B'. Joining $A'B'$ we have the polar of P' verified by $CM' = \frac{R^2}{CP'} = \frac{16}{2} = 8$. For P'' $(x'y')$ we have $x' = 16 \times 0.707 \pm 4 \times 0.707 \sqrt{16-1} = 22.225$ or 0.369. These determine the points A'', B''. The line joining these points $A''B''$ is the polar of P'', as proved by $CM'' = \frac{R^2}{CP''} = \frac{16}{1} = 16$.

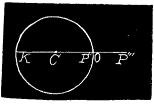
The locus of $x''y''$, while P or $x'y'$ moves along the same radius, is an equilateral hyperbola, concentric with the given circle and having its vertex at the circumference.

The proof is as follows: The co-ordinates of contact of tangents to a hyperbola, when equilateral and its semi-axes each equal to R, are $x' = \frac{R^2 x' \mp R y' \sqrt{R^2 - (x'^2 - y'^2)}}{x'^2 - y'^2}$ $y'' = \frac{R^2 y' \pm R x' \sqrt{R^2 - (x^2 - y'^2)}}{x^2 - y'^2}$. CM is now the axis of X's, and since $P(x'y')$ moves along this line, $y' = 0$. The formulas give for P, $x' = 3$, $x'' = 5\frac{1}{3}$ or CM. $y'' = \pm 3.52$ or MA and MB, which determine the points A, B. For $P' = x' = 2$, $x'' = 8$ or CM', $y'' = \pm 6.92$ or $M'A'$ and $M'B'$. For P'', $x' = 1$, $x'' = 16$, or CM'', $y'' = \pm 15.48$ or $M''A''$ and $M''B''$, all the same points as by the first formulas. Thus it appears that the *imaginary* tangents to a circle are *real* tangents to a corresponding hyperbola.

An attempt will now be made to justify the above results without recourse to the disputable expedient of changing the signs of the expression under the radical.

It is now understood that the difference between positive and negative quantities is merely the difference of counting in opposite directions, both equally real. Guided by this hint, if we let (a^2) stand for the expression under the radical sign, and consider that $(-a^2)$ is composed of factors, one of which is affected with the sign $(-)$, then, when $\sqrt{a^2} (= a\sqrt{+1})$ becomes $\sqrt{-a^2} (= a\sqrt{-1})$, we know that a factor has changed the direction in which it was before estimated, to the opposite direction. By examination we discover where and how the change took place.

In the present case it is the reversal of direction in counting from the the point $x'y'$ to the circumference. This may be seen by factoring the expression under the radical. Taking the point P''' $x'y'$ when without the circle, $x'^2 + y'^2 - R^2 = P'''C^2 - OC^2 = (P'''C + OC)(P'''C - OC) = P'''K \times P'''O$, both factors measured in the same direction. When we reach P'' *within* the circle, we have $P''K \times P''O$, in which $P''O$ is measured in the opposite direction and of course takes the sign $(-)$, which explains the change of sign in the radical expression $(\sqrt{-a^2})$. In other words, the instant that $x'y'$ crosses the circumference, $\sqrt{-1}$ appears. Call it then the sign of that crossing and nothing else, and at once the imaginary is divested of its badge of licensed irresponsibility



and falls into the ranks of orderly realities, which alone properly constitute the exact sciences. Instead of being "an expression for an impossible operation" the $\sqrt{-1}$, in its present situation at least, calls for no operation whatever, its origin being accounted for, and its office defined just as it stands—a finger-post where before it stood an impassable barrier.

The $\sqrt{-1}$ takes its rise as a residual factor or coefficient of the term whose root has been extracted. Like the corresponding factor, $\sqrt{+1}$ it has not power to either increase or diminish the *numerical* value of the term. Why then treat it as having any numerical significance? Why not call it the co-efficient of *direction*? Whether the direction be *opposite* or *perpendicular*, say that while the symbol denotes *change* of direction the *particular* direction must in each instance, be determined by its own conditions. Although by convention $(-)$ indicates direction opposite to $(+)$, that will not prevent $\sqrt{-1}$ from indicating the same when the conditions of its appearance distinctly point that way, provided this use of the symbol involves no inconsistency with its other uses.

The conditions in the present example may be restated thus: Of the elements which enter into the radical expression, viz. R and $x'y'$, the latter is referred to the limits of the former, that is to the circumference and the centre. While $x'y'$ is without the circle, both limits lie in the same direction from it. After $x'y'$ enters the circle, the limits (centre and circumference) lie in opposite directions from it. And this change is signalized by the appearance of $\sqrt{-1}$.

If C be a fixed point and $x'y'$ a moving point, approaching C on a right line from an infinite distance, then by the application of the first formulas above, the tangents will generate, by a *continuous movement*, first the semi-circumference and then the hyperbola; showing the hyperbola to be an inverse continuation of the circle, or the circle turned inside out.

When $x'y'$ reaches the limit C the tangents find their true limits in the asymtotes, and coincide with the axes. The momentary coincidence of the tangents at the other limit—the circumference—only marks the transition from one phase of their work to the other.

The tangents, moving in obedience to a uniform law, would naturally be expected, after generating the semi-circumference, not to wander off into "untraceable labyrinths," but to follow some plain path, especially as the line joining their (imaginary) points of contact, may, at any stage, be definitely located.

OUTLINE OF INVESTIGATION FOR ASYMPTOTES.

By E. S. LOOMIS, A. M., Ph. D., Professor of Mathematics in Baldwin University, Berea, Ohio.

I. Methods.

A. By Inspection.

1. If when $x\doteq 0$, $y\doteq\infty$, $x=0$ is an asymptote.
2. " " $x\doteq 0$, $y\doteq a$, or $\sqrt{-a}$, $x=0$ is not an asymptote.
3. " " $x\doteq\pm a$, $y\doteq\pm\infty$, $x=\pm a$ is an asymptote.
4. " " $x\doteq\pm a$, $y\doteq\pm a$, or $\pm\sqrt{-a}$, $x=\pm a$ is not an asymptote.
5. Treat y in some manner as x.

Note 1. To be universally true, the equation must be solved for either y or x.

Note 2. The finite quantity is seen in the curve.

B. By direct Investigation.

1. Through Intercepts.

(a). If X and Y, either or both, are finite, there is an asymptote.

(b). If X and Y are both infinite, no asymptotes.

(c). To find the equation of the asymptote, substitute the intercepts of X and Y for a and b in $\frac{x}{a}+\frac{y}{b}=1$.

Note. Under (a), if need be, find the limit of $\frac{y}{x}$.

2. By writing for y, $kx+r$, in the curve, expanding, arranging according to the descending powers of x, writing the coefficients of the two highest powers of x equal to 0, from which find the values of k and r, which values substituted in last two terms of arranged equation just found give the equation of an asymptote.

3. By solving the equation for y and developing by Maclaurin's formula, etc.

II. An example of application and illustration. I shall take the cissoid (of Diocles), because I have never seen it worked, either by text or student, except by the method of inspection. Of course by inspection, by 3rd under A above it is instantly seen that $x=2a$ is an asymptote. I have had students declare that it could not be solved by direct investigation. It comes directly under (a) of B above. First, as in Analytic Geometry, investigate for limits of the curve. We discover it is limited by $2a$ to the right. From $y^2=\frac{x^3}{2a-x}$,

$\frac{dx}{dy}=\frac{2y(2a-x)}{3x^2+y^2}$ second in $X=x-y\frac{dx}{dy}$, sub. value of y and $\frac{dx}{dy}$, gives

$$x=\frac{ax^3}{3ax^2-x^3}=\frac{ax}{3a-x}=\frac{2a^2}{a}=2a, \text{ since } x=2a \text{ at limit.}$$

Concluded on page 204.

ARITHMETIC.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTION OF PROBLEMS.

47. Proposed by F. P MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, in New Windsor College, New Windsor, Maryland.

Mr. Merchant sells 20% above cost, with weights and measures $12\frac{1}{2}\%$ "short," allows a discount of \$5 on every bill of \$50, and loses 5% of his sales as "bad debts." Find his *rate per cent.* of net profit, or net loss; one cent in every dollar of sales proves counterfeit, and collection-charges are $2\frac{1}{2}\%$.

I. Solution by H. W. DRAUGHON, Olio, Mississippi. and the PROPOSER.

Let $\$C$=the cost of the merchandise sold; then $\frac{120}{100}$ of $\frac{8}{7}$ of $\$C=\frac{48}{35}$ of $\$C$=the amount of the merchandise sold. As per the problem, the *aggregate* of deductions to be made from the *amount of the sales* is $18\frac{1}{2}\%$; that is, the *net* amount of the sales is $\frac{978}{875}$ of $\$C$, and the *net* profit is $\frac{103}{875}$ of $\$C$. Hence the required rate per cent. of *net* profit must be $11\frac{27}{35}$.

II. Solution by G. B. M. ZERR, A. M., Principal of High Schools, Staunton, Virginia.

$100\%-12\frac{1}{2}\%=87\frac{1}{2}\%$, what he sells for 120%.

$120\%\div87\frac{1}{2}=137\frac{1}{7}\%$, what he gets for 100%.

\$5 on \$50=10%.

$10\%+5\%+1\%+2\frac{1}{2}\%=18\frac{1}{2}\%$, what he loses.

$137\frac{1}{7}\times18\frac{1}{2}=25\frac{13}{35}\%$.

$137\frac{1}{7}-25\frac{13}{35}=111\frac{27}{35}\%$. $\therefore$ he gains $11\frac{27}{35}\%$.

Also solved by *P. S. BERG*.

PROBLEMS.

52. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College New Windsor, Maryland.

By selling a horse for H=\$150 cash, I gain $p=20\%$. At what price should I sell the horse and wait $d=90$ days, money worth $m=6\%$, in order to gain $q=25\%$?

53. Proposed by P. S. BERG, Apple Creek, Ohio.

\$500 Wooster, O., Sept. 2nd, 1886.

One year after date we, or either of us promise to pay to the order of J. M. W. Five Hundred Dollars for value received with 7% annual interest from date.

J. C.
M. C.

Endorsed May 13, 1893, \$75.00
" Sept. 1, 1894, \$300.00.

What was due April 1st, 1895?

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTION OF PROBLEM.

43. **Proposed by F. M. SHIELDS, Coopwood, Mississippi.**

Four men, A, B, C, and D, start from the same place, the traveling rates of A and C are equal, and the traveling rates of B and D were as 17 to 18, respectively; B could travel one mile in 7 minutes and 12 seconds. A traveled *due* west a certain distance, B traveled *due* north the cube of A's distance *plus* his distanc; C traveled *due* east a certain distance, and D traveled *due* south the cube of C's distance *plus* his distance: They all change directions, and A traveled *due* north a certain distance, B traveled *due* east the 5th power of A's distance north; C traveled *due* south a certain distance, and D traveled *due* west the 5th power of C's distance south,—when it was found that the sum of the north and south distances traveled by B and D was 351090 feet, and the sum of the distances that B and D traveled east and west was 5939200000 feet, and that the product of the distances that A and C traveled east and west *plus* the square of the difference of these distances, *plus one* was 3901: and that the product of the distances that A and C traveled north and south *plus* the square of the difference *squared*, plus the product multiplied by the square of the difference, was 494100.00 [equal to the following new formulas: $(nn+d^2+1)=3901$, and $\{(nn+d^2)^2+(nn\times d^2)\}=49410000$]. How far on a line is each party from the starting place, and how long did it require for B and D each to make the entire trip from starting place to the end?

Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let x=distance A travels due west
y= " C " " east
z= " A " " north
u= " C " " south.

Then x^3+x=distance B travels due north
y^3+y= " D " " south
z^5= " B " " east
u^5= " D " " west

Also $x^3+y^3+x+y=351090$......(1)

$z^5+u^5=5929200000$......(2)

$xy+(x-y)^2+1=3901$......(3)

$\{zu+(z-u)^2\}^2+zu(z-u)^2=49410000$......(4)

From (3), $x^2-xy+y^2+1=3901$......(5)

From (1), $(x+y)(x^2-xy+y^2+1)=351090$......(6)

(5) in (6) gives, $x+y=90$......(7)

(7) in (1) gives, $x^3+y^3=351000$......(8)

(7) cubed minus (8) gives, $3xy(x+y)=378000$......(9)

(7) in (9) gives, $xy=1400$......(10)

From (7) and (10), $x=70$ or 20, $y=20$ or 70

From (2), $(z+u)(z^4-z^3u+z^2u^2-zu^3+u^4)=5929200000$......(11)

From (4), $z^4-z^3u+z^2u^2-zu^3+u^4=49410000$......(12)

(12) in (11) gives, $z+u=120$......(13)

$[\{(13)\}^5-(2)]\div5$ gives, $zu(z^3+2z^2u+2zu^2+u^3)=3790800000$......(14)

$\{(13)\}^3$ in (14) gives $zu(1728000-z^2u-zu^2)=3790800000$......(15)

(13) in (15), after reduction, gives, $z^2u^2-14400zu=-31590000$

$\therefore zu=11700$ or 2700......(16).

From (13) and (16) we get, $z-u=\pm60$ or $\pm180\sqrt{-1}$.

$\therefore z=90$ or 30 or $30(2\pm3\sqrt{-1})$,

$u=30$ or 90 or $30(2\mp3\sqrt{-1})$.

A is distant from the starting point either of the following:—

$\sqrt{90^2+70^2}=114.017+$feet, $\sqrt{90^2+20^2}=92.196$ feet.

$\sqrt{30^2+70^2}=76.157$ feet, $\sqrt{30^2+20^2}=36.055$ feet.

C is distant from the starting point either of the following:—

36.055 feet, 76.157 feet, 92.196 feet, 114.017 feet.

B is distant from the starting point either of the following:—

$\sqrt{(70^3+70)^2+(90^5)^2}=5904900009.965$ feet,

$\sqrt{(20^3+20)^2+(90^5)^2}=5904900000.005$ feet,

$\sqrt{(70^3+70)^2+(30)^{10}}=24302421.629$ feet,

$\sqrt{(20^3+20)^2+(30)^{10}}=24300001.323$ feet.

D is distant from the starting point either of the following:—

24300001.323 feet, 24302421.629 feet, 5904900000.005 feet, 5904900009.965 feet.

B has traveled either of the following distances:

$70^3+70+90^5=5905243070$ feet$=118417.38447$ miles,

$20^3+20+90^5=5904908020$ feet$=118353.79167$ miles,

$70^3+70=30^5=24643070$ feet$=4667.24811$ miles,

$20^3+20+30^5=24308020$ feet$=4603.79167$ miles.

D has traveled either of the following distances:

4603.79167 miles, 4667.24811 miles, 118353.79167 miles, 118417.38447 miles.

B travels 1 mile in 7 min. 12 sec.$=432$ sec.

D travels one mile in $\frac{17}{18}$ of $432=408$ sec.

It has taken B either of the following times:—

$118417.38447\times432=51156310.09$ sec.$=14210$ h. 5 m. 10.09 sec.,

$118353.79167\times432=51128838$ sec.$=14202$ h. 27 m. 18 sec.,

$4667.24811\times432=2016251.18$ sec.$=560$ h. 4 m. 11.18 sec.,

$4603.79167\times432=1988838$ sec.$=552$ h. 27 m. 18 sec.

It has taken D either of the following times:—

$4603.79167\times408=1878347$ sec.$=521$ h. 45 m. 47 sec.,

$4667.24811\times408=1904237.22$ sec$=528$ h. 57 m. 17.22 sec.,

$118353.79167\times408=48288347$ sec.$=13413$ h. 25 m. 47 sec.,

$118417.38447\times408=48314292.86$ sec.$=13420$ h. 38 m. 12.86 sec.

The imaginary results have been omitted as impossible.

The above solution was the first received and entitles *Prof. ZERR* to the St. Andrews, Fla., City Lot, offered by the *PROPOSER*. Excellent solutions of later date were received from *H. C. WHITAKER*, *M. A. GRUBER*, *H. W. DRAUGHON*, *H. C. WILKES*, *A. H. BELL*, *A. L. FOOTE*, *A. H. HOLMES*, and *P. S. BERG*.

PROBLEMS.

52. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

In how many ways can we arrange 12 friends of the MONTHLY, around a table, so that; (1), the editors may never be together. (2), Matz and Halsted may never be apart; and (3), Zerr and Ellwood may always have Gruber betwixt them?

53. **Proposed by LEONARD E. DICKSON, M.A., Fellow in Mathematics, University of Chicago.**

Can it be proven that the value of the expression

$$\left\{ 5x-\frac{5x+3}{3}\cdot\frac{5x(5x-1)}{1.2}+\frac{(5x+3)(5x+8)}{4.5}\cdot\frac{5x(5x-1)(5x-2)}{1.2.3} \right.$$
$$-\frac{(5x+3)(5x+8)(5x+13)}{5.6.7}\cdot$$
$$\frac{5x(5x-1)(5x-2)(5x-3)}{1.2.3.4}+\frac{(5x+3)\ldots(5x+18)}{6.7.8.9}\cdot\frac{5x\ldots\ldots(5x-4)}{1\ldots5}+\ldots$$
$$\left. +(-1)^{5x-2}\frac{(5x+3)\ldots(30x-12)}{5x\,(5x+1)\ldots(10x-3)}\cdot 5x+(-1)^{5x-1}\frac{(5x+3)\ldots(30x-7)}{(5x+1)\ldots(10x-1)} \right\}$$

is identically zero?

GEOMETRY.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEM.

42. **Proposed by G. I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, New Hampshire.**

If the bisectors of two angles of a triangle are equal the triangle is isosceles.

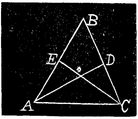
II. **Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia; H. C. Whitaker, M. S., M. E., Professor of Mathematics. Manual Training School, Philadelphia, Pennsylvania; WILLIAM HOBBY, a Student in University of Tennessee, Knoxville, Tennessee.**

Let $AC=b$. Then $AD=b\sin C/\sin(C+\frac{1}{2}A)$,

$CE=b\sin A/\sin(A+\frac{1}{2}C)$.

$\therefore \sin C\sin(A+\frac{1}{2}C)=\sin A\sin(C+\frac{1}{2}A)$.

Let $x=\sin\frac{1}{2}A$, $y=\sin\frac{1}{2}C$.



Then $(2xy-2x^3y-y^2+2x^2y^2)\sqrt{1-y^2}$

$=(2xy-2xy^3-x^2+2x^2y^2)\sqrt{1-x^2}$.

$\therefore (x^2-y^2)\{x^2+y^2-x^4-x^2y^2-y^4-4xy$
$+4x^3y+4xy^3-4x^3y^3\}=0$.

$\therefore x=y$ and $\angle A=\angle C$,

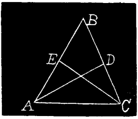
also, $(x-y)^4-(x-y)^2+xy(1-xy)+xy(1-2xy)^2=0$.

III. **Solution by J. H. GROVE, Howard Payne College, Brownwood, Texas.**
(Solution by Reductio ad Absurdum.)

Suppose the $\triangle$ to be scalene and that $AB > BC$.

(1) $AB.AC = BD.DC + AD^2$

(2) $BC.AC = BE.AE + CE^2$



Then (1) $AB.AC - BD.DC = AD^2$ and

(2) $BC.AC - BE.AE = CE^2$. But $AD^2 = CE^2$. Hyp.

$\therefore$ $AB.AC - BD.DC = BC.AC - BE.AE$.

But $AB.AC > BC.AC$ (Hyp.), and

$BD.DC < BE.AE$ ($\angle a < \angle c$ Hyp.)

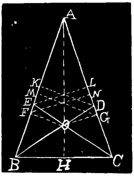
($BD < BE$ and $DC < AE$)

$\therefore$ The conclusion above reached: $AB.AC - BD.DC = BC.AC - BE.AE$ is absurd. It can be true only in case the $\triangle$ is isosceles.

$\therefore$ If the bisectors AD and CE are equal, $AB = BC$. *Q. E. D.*

IV. **Solution by EDW. R. ROBBINS, Master in Mathematics of Lawrenceville School, Lawrenceville, New Jersey.**

Let ABC be a $\triangle$ of which EC and BD are equal bisectors of base angles. To prove the $\triangle$ is isosceles. Draw third bisector AO. Draw perpendiculars OH, OF, OG: these are equal lines. From E, D, F, G, draw perpendiculars to opposite side.



In $\triangle$s AFO, AOG: $FO = OG$: $AO = AO$ and angles at A are equal.

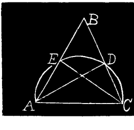
Therefore these triangles are equal right triangles, and $AF = AG$. In right $\triangle$s AGM and AFN angle at A is common and $AF = AG$. Hence $\triangle$s are equal and $FN = GM$.

Now the right triangles AEL and ADK are similar respectively to AFN and AGM. But these last $\triangle$'s are equal and hence the $\triangle$'s AEL and ADK are equal. And therefore $AE = AD$.

The triangles ADB and AEC have two sides of the one, AD, DB equal respectively to two sides of the other AE, EC, also the angle at A is common. Hence the $\triangle$s are equal. Therefore $AB = AC$ and original $\triangle$ ABC is isosceles. *Q. E. D.*

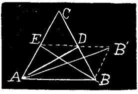
V. **Solution by P. S. BERG, Apple Creek, Ohio.**

Let ABC be a triangle, $AD = CE$. Through the three points A, C and E pass the circumference of a circle. It will also pass through D. For if it meets AD ni P between A and D the arc EA must be greater than PE, since ECA which is equal to D CE is greater than PCE. Also the arc PE is equal to the arc PC, since the angle EAP is equal to the angle PAC. Whence the arc AEP is greater than



EPC, and consequently the chord AP is greater than the chord CE. But by hypothesis $CE=AD$; then AP is greater than AD which is impossible. Hence the circle which passes through A, C, and E cannot cut AD between A and D. In like manner it can be shown that the circle cannot cut AD beyond D. Hence it must pass through D. Hence the angle EAD which is half of A is equal to DCE which is half of C. Therefore the angle A is equal to the angle B and the triangle is isosceles. *Q. E. D.*

VI. Solution by W. W. MOSS, Instructor in Mathematics, Brown University, Providence, Rhode Island.



Let AD and BE be equal bisectors of the $\angle$s CAB and CBA of the $\triangle ABC$. To prove $\triangle ABC$ isosceles. Move the $\triangle DBA$ so that the side AD will coincide with its equal EB, A falling at E, D at B. Then fold $\triangle BAD$ upon AD (EB) as axis till B falls upon the plane at B', $\triangle BAD$ taking position EBB'. Draw AB'. Consider the $\triangle$s AEB' and ABB' $\angle AEB'=\angle AEO+\angle OEB'$ $=\angle AEO+\angle OAB=\angle AEO+\angle OAE=\angle AOB$ $\angle ABB'=\angle ABE$ $+\angle EBB'=\angle OBD+\angle ODB=\angle AOB$. $\therefore$ $\angle AEB'=\angle ABB'$ $\angle CAB$ $+\angle CBA<180^\circ$ and halving $\angle OAB+\angle OBA<90^\circ$. $\therefore$ $\angle AOB=180^\circ$ $-(\text{angle}\,OAB+\text{angle}\,OBA)>90^\circ$. $\therefore$ angles AEB' and ABB' are obtuse and equal side $EB'=$side AB and $AB'=AB'$. $\therefore$ $\triangle AEB'=\triangle ABB'$ having two sides and an opposite angle in one equal to homol. parts in the other, the equal angles being obtuse. $\therefore$ angle $EAB=$angle $EAB'+$angle $BAB'=$angle $AB'B$ $+$angle $AB'E=$angle $EB'B=$angle DBA.

$\therefore$ $\triangle ABC$ is isosceles. *Q. E. D.*

Solutions of this problem were received from *J. C. CORBIN, WILLIAM PARKISON, F. P. MATZ, O. W. ANTHONY, A. M. HUGHLETT, H. W. DRAUGHON and J. F. W. SCHEFFER. Professor SCHEFFER sent in three solutions and Professor GROVE two.*

Note.—An excellent demonstration of this proposition is given on page 44, of *Dr. Halsted's Elementary Synthetic Geometry.*

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

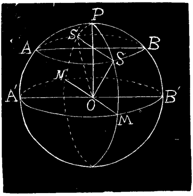
32. Proposed by J. F. W. SCHEFFER, Hagerstown, Maryland.

Suppose it to be possible to perform the passage through the north pole: at

what latitude would the maximum distance be saved by a ship sailing on the arc of a great circle instead of a parallel of latitude, the points of departure and destination being 180° apart? Also find the maximum saving.

I. Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland, and E. D. SCALES, Student of Junior Class, University of Mississippi, University, Mississippi.

Put $\angle MOS=\lambda$, $=$the latitude of the *departure-point* of the ship.



According to the problem, the ship departing from S may reach the destination S' by taking either the *longer* route SAS' $=(\pi\cos\lambda)R$, or by taking the *shorter* route $SPS'=(180°-2\lambda)R$. The expression for the number of miles saved by taking the route SPS', becomes $M=2(\lambda-\pi\sin^2\frac{1}{2}\lambda)R$....(1), which is to be a maximum. Differentiating, etc., $\lambda=\sin^{-1}(2/\pi)$, $=39°\ 32'\ 24''.784$; and, consequently, the number of miles saved is $M=[2\sin^{-1}(2/\pi)-\pi$ $+\sqrt{(\pi^2-4)}]R$, $=2620.80359+$ English miles.

II. Solution by O. W. ANTHONY, M. S., Missouri Military Academy, Mexico, Missouri, G. B. M. ZERR, A. M., Staunton, Virginia, and the PROPOSER.

Let λ be the latitude and regard the earth as a perfect sphere with the same volume.

Then the radius of the small circle in latitude λ is $R\cos\lambda$, where $R=$radius of the earth.

$\therefore\ \pi R\cos\lambda$ is the distance to sail on a parallel of latitude and $(\pi-2\lambda)R$, the distance on a great circle.

$\therefore\ \pi R\cos\lambda-(\pi-2\lambda)R=\text{max.}$

$\therefore\ \pi\cos\lambda-\pi+2\lambda=\text{max.}$

Integrating, we get $\lambda=\sin^{-1}\dfrac{2}{\pi}=39°\ 32'\ 24''.55$. The saving $=R(\pi\cos\lambda-\pi+2\lambda)$. But $R=20902410$ feet according to Col. Clark.

$\therefore$ the saving$=20902410\times.671385=14033564.53785$ feet

$=2657.87207$ miles.

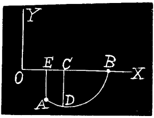
Also solved by *A. H. BELL, Hillsboro, Illinois*, and *E. W. MORRELL, Montpelier Seminary, Montpelier, Vermont.*

33. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, California.

Show that of all curves of a given length, traced on one plane between two given points, and are made to revolve around a common axis situated in that plane, the Catenary generates a minimum area.

I. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let A, B be the two given points, arc $ADB=S$, OBX the common



axis, $OB=a$, $OE=b$, $OC=x$, $CD=y$. Then from Calculus of Variations, we get:—

$$\int Vdx=\int_b^a 2\pi y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx=\text{a minimum}$$

$$\text{and}\int V'dx=\int_b^a\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx=S.$$

$$\therefore DU=D\int(V+\beta V')dx=0.$$

$$V+\beta V'=(2\pi y+\beta)\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}},=\text{also } P_1\frac{dy}{dx}+c$$

$$\text{where } P_1=(2\pi y+\beta)\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-\frac{1}{2}}\frac{dy}{dx}.$$

$$\therefore C+(2\pi y+\beta)\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-\frac{1}{2}}\left(\frac{dy}{dx}\right)^2=(2\pi y+\beta)\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}.$$

$$\therefore dx=\frac{cdy}{\sqrt{(2\pi y+\beta)^2-c^2}}.\quad\text{Let } 2\pi y+\beta=z,\text{ then } dy=\frac{dz}{2\pi}.$$

$$\therefore dx=\frac{c}{2\pi}\frac{dz}{\sqrt{z^2-c^2}},\quad x=\frac{c}{2\pi}\log\frac{C'}{z-\sqrt{z^2-c^2}}=\frac{c}{2\pi}\log\frac{C(z+\sqrt{z^2-c^2})}{c^2}.$$

$$\therefore x=\frac{c}{2\pi}\log\frac{C\{2\pi y+\beta+\sqrt{(2\pi y+\beta)^2-c^2}\}}{c^2}.$$

$$\therefore y=\frac{1}{4\pi C}\left(c^2e^{\frac{2\pi x}{c}}+C^2e^{-\frac{2\pi x}{c}}\right)-\frac{\beta}{2\pi},\text{ the equation of a Catenary.}$$

II. Solution by J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering, A. and M. College, College Station, Texas.

It is shown in Statics that of all curves of given length between two given points the Catenary has its center of gravity lower than any other—the curve being converse to the axis, and gravity acting at right angles thereto (See Minchin's Statics, Vol. II, page 161).

Now by the theorem of Pappus the area generated equals the length of curve into path described by the center of gravity in turning about the axis; and since the radius of the circle described by the *c. g.* of the Catenary is less than for any other curve considered it is evident the area generated is a minimum.

If the Catenary is concave to the axis of x the area generated will be a maximum, and is so proven in Duhamel's "Elements de Calcul Infinitesimal, Tome Second", page 400.

Prof. MATZ, sent three excellent solutions to the above problem.

PROBLEM.

42. **Proposed by J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering, A. and M. College, College Station, Texas.**

Show that the volume included between the surface represented by the equation $z=e^{-(x^2+y^2)}$ and the xy plane equals the square of the area of the section made by the zx plane, the limits of x and y being plus and minus infinity.

MECHANICS.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

20. **Proposed by CHAS. E. MYERS, Canton, Ohio.**

A flexible cord of given length is suspended from two points whose co-ordinates are $(x, y,)$ and (x', y'). What must be the condition of the cord in order that it may hang in the arc of a circle?

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Taking the lowest point for origin, and the horizontal and vertical lines through it for axes of x and y, and $s=a\varphi$......(1) for the intrinsic equation to the circle.

If $\pi=$ the constant horizontal component of tension at all points of the cord, the law of mass as given by Theoretical Mechanics is

$m=\frac{\pi}{g}\frac{d^2y}{dx^2}\Big\backslash\frac{ds}{dx}$....(2). We have $\frac{dy}{dx}=\tan\varphi$, $\frac{dx}{ds}=\cos\varphi$, $\frac{ds}{d\phi}=a$, from (1);

$$\frac{d^2y}{dx^2}=\frac{1}{\cos^2\varphi}\frac{d\varphi}{dx}=\frac{1}{\cos^2\varphi}\frac{d\varphi}{ds}\frac{ds}{dx}=\frac{1}{a\cos^3\varphi}......(3).$$

Then (2) gives $m=\frac{\pi}{g}\frac{a}{a^2\cos^2\varphi}=\frac{\pi a}{(a-y)^2}$....(4), or the mass unit varies inversely as the square of the distance below the horizontal diameter.

Excellent solutions of this problem were also received from *G. B. M. ZERR, and F. P. MATZ.*

21. **Proposed by J. A. CALDERHEAD, Superintendent of Schools, Limaville, Ohio.**

Show that, in the wheel and axle, when a force P, acting at the circumference of the wheel, supports a weight Q upon the axle,

$$P(R\mp\rho\sin\varepsilon)=Q(r\pm\rho\sin\varepsilon)\pm W\rho\sin\varepsilon,$$

where R, r, and ρ are the radii of the wheel, the axle, and their common axis respectively, and ε is the limiting angle of resistance.

Solution By G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia; and F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Let W=weight of wheel and axle, β=the angle between P and Q and also between P and W since Q and W are parallel. The resultant of P, Q, W due to friction is $\pm\sqrt{(Q+W)^2+P^2+2P(Q+W)\cos\beta}\times\rho\sin\varepsilon$.

$\therefore PR=Qr\pm\rho\sin\varepsilon\sqrt{(Q+W)^2+P^2+2P(Q+W)\cos\beta}$.

When $\beta=0$ this becomes $PR=Qr\pm\rho\sin\varepsilon(P+Q+W)$.

$\therefore P(R\mp\rho\sin\varepsilon)=Q(r\pm\rho\sin\varepsilon)+W\rho\sin\varepsilon$.

Also solved by *ALFRED HUME.*

22. Proposed by DE VOLSON WOOD, C. E., Professor of Mechanical and Electrical Engineering in Stevens Institute of Technology, Hoboken, New Jersey.

A prismatic bar having a uniform angular velocity ω and a linear velocity of v feet per second, suddenly snaps (by the disappearance of the cohesive force) into an indefinite number of equal parts: required the resultant angular velocity of each piece and the locus of the parts at the end of t seconds after rupture.

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi.

Take O, the center of gravity of the bar AB, as the origin of a system of rectangular axes, the Y—axis coinciding with the direction of the motion of translation.

Let the motion of rotation be contrary to that of the hands of a clock.

Let the length of the bar be $2nl$, n being the number of equal parts into which it snaps; and let the cross-section and the density, each, be unity.

Denote the middle point of DE, any one of these equal parts, by C, any other point of DE by P.

Let $OC=R$, $OP=r$, and $\angle xOP=\theta$.

At the instant of separation P has a velocity, v, parallel to Y and a velocity, $r\omega$, perpendicular to OB.

The subsequent motion of DE may be determined by supposing it initially at rest and acted upon by such impulsive forces as are expressed in the actual motion at the instant under consideration.

The element of mass at P is acted upon by an impulsive force parallel to Y measured by the momentum $v.dr$, and by a force perpendicular to OB measured by $r\omega.dr$.

Therefore, taking moments about C, the angular velocity of DE, given by the ratio of the moment of the momentum to the moment of inertia, is

$$\frac{\int[v.\cos\theta+r\omega](r-R)dr}{\frac{2}{3}l^3}\text{ the limits being } R+l \text{ and } R-l.$$

Integrating between these limits, the numerator of this fraction becomes $\frac{2}{3}\omega l^3$.

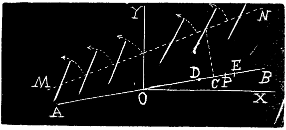
Hence, after separation, DE will rotate about C with an angular velocity equal to that of the original bar.

C, itself, will move in the direction OY with a velocity $v+R\omega.\cos\theta$ and in the direction XO with a velocity $R\omega.\sin\theta$.

At the end of t seconds the co-ordinates of C will be given by $x=R\cos\theta-R\omega.\sin\theta.t$ and $y=R\sin\theta+(v+R\,\omega.\cos\theta t)$.

Eliminating R, $y=\dfrac{\tan\theta+\omega t}{1-\omega t.\tan\theta}\,.\,x+vt$. or $y=\tan(\theta+\tan^{-1}\omega t)x+vt$.

The locus of the centers of gravity of the parts t seconds after rupture is, therefore, a straight line inclined to X at an angle $\theta+\tan^{-1}\omega t$, and cutting Y at a distance vt from O.



This line coincides with Y after $\dfrac{\cot\theta}{\omega}$ seconds.

When $t=\infty$, the locus is perpendicular to AB.

In the figure the line MN represents the locus, and the arrows the direction of rotation of the parts. The center of gravity of each part moves uniformly in a straight line forever, while the part rotates uniformly about this center of gravity.

This problem was also solved by *F. P. MATZ.*

PROBLEMS.

29. Proposed by J. A CALDERHEAD, A. B., Superintendent of Schools, Limaville, Ohio.

Show that if a body be projected from the angle A of a plane triangle ABC so as to strike the side CB at a point D, then, if its course after reflection at D be parallel to AB, $\tan DAB=\dfrac{(1+\varepsilon)\cot B}{(1-\varepsilon)\cot^2 B}$.

30. Proposed by WILLIAM HOOVER. A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

P is the lowest point on the rough circumference of a circle in a vertical plane at which a partical can rest, friction being equal to the pressure; to find the inclination of the radius through P to the horizon.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

23. Proposed by J. M. COLAW, A. M., Principal of High School, Monterey, Virginia.

Find three positive integral numbers such that the product of the first and the sum of the others is a square and the sum of their cubes is a square.

Solution by J. W. NICHOLSON, A. M., LL. D, President and Professor of Mathematics in the Louisiana State University, and Agricultural and Mechanical College, Baton Rouge, Louisiana.

Let x^2, $2(n^2-1)x^2$, $2(n^2+1)x^2$, be the three integers. The first condition gives $4n^2x^4=\square$; hence we have only to solve

$$\left[x^2\right]^3+\left[2(n^2-1)x^2\right]^3+\left[2(n^2+1)x^2\right]^3=\square,$$

$$\text{or } x^6\left[16n^6+48n^2+1\right]=\square.$$

$\therefore\ 16n^6=\left(\frac{48}{2}n^2\right)^2$; whence $n^2=36$; and the three integers are x^2, $70x^2$ and $74x^2$, where x is any integer.

Also solved by *H. W. DRAUGHON*, *and G. B. M. ZERR*.

24. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Solve *generally:* The sum of the cubes of n consecutive numbers is a square. Determine the numbers, when $n=2$, $n=3$, $n=4$, and $n=5$.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let m, $m+1$, $m+2$, $m+3$, etc., represent any consecutive numbers the sum of whose cubes is to be taken.

Solving by the differential method, we obtain

$$S=\frac{n}{4}\left\{n^3+(4m-2)n^2+(6m^2-6m+1)n+4m^3-6m^2+2m\right\}.$$

This reduces to $S=\frac{1}{4}\left\{[n(n+2m-1)]^2+2m(m-1)[n(n+2m-1)]\right\}$, (A).

If the sum of the consecutive cubes is to be a square, then

$$[n(n+2m-1)]^2+2m(m-1)[n(n+2m-1)]=\square=a^2.$$

Adding $[m(m-1)]^2$ to both members, we have

$$[n(n+2m-1)+m(m-1)]^2=a^2+[m(m-1)]^2.$$

This is of the form $(p^2+q^2)^2=(2pq)^2+(p^2-q^2)^2$.

Equating the respective values, and reducing for m and n, we obtain $2m=1+\sqrt{4p^2-4q^2+1}$, and $2n=\sqrt{4p^2+4q^2+1}-\sqrt{4p^2-4q^2+1}$. It will be observed that each of the radical quantities is of the form of an odd square, $4p+1$.

There are two conditions that will render the radicals rational, and, at the same time, have a an integer:—

(1) When $4q^2=4p$. Then $m=p=q^2$, and $n=1$. According to this condition there is but *one cube* that can be taken at one time, and hence there would be no *sum of cubes*. This cube is the cube of a square, and is, therefore, also the square of a cube.

(2) The second condition is when $p^2=q^2$. Then $m=1$; and substituting this value in (A), we obtain $S=\left\{\frac{n(n+1)}{2}\right\}^2$, which is the square of the sum of the series, $1+2+3+\ldots\ldots+n$. From this, then, we have $1^3+2^3+3^3+\ldots.+n^3=(1+2+3+\ldots.+n)^2$, or *the square of the sum of the first n natural numbers is equal to the sum of their respective cubes.*

Therefore, in order that the sum of the cubes of n consecutive numbers be a square, *the first number must be unity.*

When $n=2$, the numbers are 1 and 2; when $n=3$, the numbers are 1, 2, and 3; &c., &c.

Also solved by *Professor COOPER D. SCHMITT, and the PROPOSER.*

II. **Solution by B. F. FINKEL, A. M., Professor of Mathematics in Kidder Institute, Kidder, Missouri.**

Let $S=1+2+3+\ldots\ldots+n=(n+1)n/2$;

$S_2=1^2+2^2+3^2+\ldots\ldots+n^2=n(n+1)(n+2)/6$; and

$S_3=1^3+2^3+3^3+\ldots\ldots+n^3=?$

$$
\begin{aligned}
\text{Now } (n+1)^4-n^4 &= \qquad\qquad\qquad\qquad\qquad\qquad 4n^3+6n^2+4n+1\\
n^4-(n-1)^4 &= 4n^3-6n^2+4n-1=4(n-1)^3+6(n-1)^2+4(n-1)+1\\
(n-1)^4-(n-2)^4 &= 4n^3-18n^2+28n-15=4(n-2)^3+6(n-2)^2+4(n-2)+1\\
(n-2)-(n-3)^4 &= 4n^3-30n^2+76n-65=4(n-3)^3+6(n-3)^2+4(n-3)+1\\
\ldots\ldots &= \ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
5^4-4^4 &= 369 \quad =4\times4^3+6\times4^2+4\times4+1\\
4^4-3^4 &= 175 \quad =4\times3^3+6\times3^2+4\times3+1\\
3^4-2^4 &= 65 \quad =4\times2^3+6\times2^2+4\times2+1\\
2^4-1^4 &= 15 \quad =4\times1^3+6\times1^2+4\times1+1\\
1^4-0^4 &= 1 \quad =4\times0^3+6\times0^2+4\times0+1
\end{aligned}
$$

Adding, $(n+1)^4=4S_3+6S_2+4S+n+1$.

Whence, $S_3=[(n+1)^4-n-1-6S_2-4S]/4$,

$=[(n+1)^4-n-n(n+1)(n+2)-2(n+1)n]/4$,

$=[n^4+2n^3+n^2]/4=[\tfrac{1}{2}n(n+1)]^2$.

If $n=2$, $S_3=9$; if $n=3$, $S_3=36$; if $n=4$, $S_3=100$; if $n=5$, $S_3=225$.

NOTE.—The above method is useful in summing the series: $1^r+2^r+3^r+4^r \ldots\ldots n^r$, where r is any integer.

PROBLEMS.

32. Proposed by A. H. BELL, Hillsboro, Illinois.

Decompose into its prime factors the number 5497558138889.

33. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find three different *sixth powers* whose sum is a square.

[The solution of this problem, if possible, is an answer to the *note* under the solution of Prob. 16.]

AVERAGE AND PROBABILITY.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

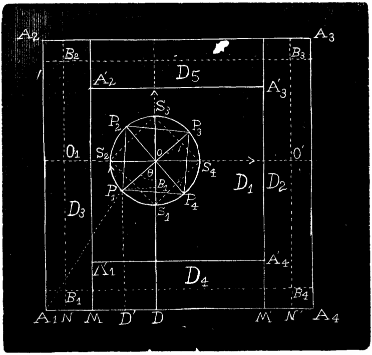
SOLUTIONS OF PROBLEMS.

20. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

A surface one inch square is thrown at random upon a surface one foot square, but in such a manner as always to lie wholly upon the larger surface. Find the mean value of *the sum of the distances* of the vertices of the smaller surface, from *any* vertex of the larger surface.

Solution by the PROPOSER.

Let $S_1S_2=s$, $A_1A_2=12s$, $AD=x$, $DO=y$; then $S_1O=\frac{1}{2}s\sqrt{2}$. Put $\angle S_1OP_1=\angle S_2OP_2=\angle S_3OP_3=\angle S_4OP_4=\theta$; then $DD'=\frac{1}{2}s\sqrt{2}\sin\theta$, and $(DO-P_1D')=\frac{1}{2}s\sqrt{2}\cos\theta$.



$\therefore\ A_1D'=\frac{1}{2}(2x-s\sqrt{2}\sin\theta)$, and $P_1D'=\frac{1}{2}(2y-s\sqrt{2}\cos\theta)$; also, $\triangle_1=A_1P_1=\frac{1}{2}\sqrt{[(2x-s\sqrt{2}\sin\theta)^2+(2y-s\sqrt{2}\cos\theta)^2]}$, $\triangle_2=A_1P_2=\frac{1}{2}\sqrt{[(2x-s\sqrt{2}\cos\theta)^2+(2y+s\sqrt{2}\sin\theta)^2]}$, $\triangle_3=A_1P_3=\frac{1}{2}\sqrt{[(2x+s\sqrt{2}\sin\theta)^2+(2y+s\sqrt{2}\cos\theta)^2]}$, $\triangle_4=A_1P_4=\frac{1}{2}\sqrt{[(2x+s\sqrt{2}\cos\theta)^2+(2y-s\sqrt{2}\sin\theta)^2]}$.

Let $\triangle=(\triangle_1+\triangle_2+\triangle_3+\triangle_4)$; then five cases confront us, for consideration.

First Case—The point O may lie in the square surface $A'_1A'_2A'_3A'_4$. If $\frac{1}{2}(24-\sqrt{2})s=a$ and $\frac{1}{2}s\sqrt{2}=b$, the mean value of the sum of the distances in this case becomes

$$\mathbf{D}_1=\int_b^a\int_b^a\int_0^{\frac{1}{2}\pi}[\triangle_1+\triangle_2+\triangle_3+\triangle_4]dxdyd\theta\div\int_b^a\int_b^a\int_0^{\frac{1}{2}\pi}dxdyd\theta\ldots.(1).$$

Second Case—The point O may lie in the rectangular surface D_2; then if $11\frac{1}{2}s=c$, $\frac{1}{2}s=e$, and $\sin^{-1}(x/e\sqrt{2})=\phi$, the mean value in this case becomes

$$\mathbf{D}_2=\int_a^c\int_e^c\int_0^{\phi}[\triangle_1+\triangle_2+\triangle_3+\triangle_4]dxdyd\theta\div\int_a^c\int_e^c\int_0^{\phi}dxdyd\theta\ldots.(2).$$

Third Case—The point O may lie in the rectangular surface D_3; and in this case,

$$\mathbf{D}_3=\int_e^b\int_e^c\int_0^{\phi}[\triangle_1+\triangle_2+\triangle_3+\triangle_4]dxdyd\theta\div\int_e^b\int_e^c\int_0^{\phi}dxdyd\theta\ldots.(3).$$

Fourth Case—The point O may lie in the rectangular surface D_4. Put $\frac{1}{2}s(23-\sqrt{2})=f$, and $\frac{1}{2}s(\sqrt{2}+1)=g$; then the mean value in this case becomes

$$\mathbf{D}_4=\int_g^f\int_e^b\int_0^{\phi}[\triangle_1+\triangle_2+\triangle_3+\triangle_4]dxdyd\theta\div\int_g^f\int_e^b\int_0^{\phi}dxdyd\theta\ldots.(4).$$

Fifth Case—The point O may lie in the rectangular surface D_5; then the mean value in this case becomes

$$\mathbf{D}_5=\int_g^f\int_a^c\int_0^{\phi}[\triangle_1+\triangle_2+\triangle_3+\triangle_4]dxdyd\theta\div\int_g^f\int_a^c\int_0^{\phi}dxdyd\theta\ldots.(5).$$

Consequently the *required* mean value becomes

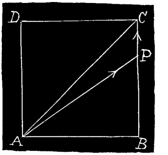
$$\mathbf{D}=\tfrac{1}{5}(\mathbf{D}_1+\mathbf{D}_2+\mathbf{D}_3+\mathbf{D}_4+\mathbf{D}_5)\ldots.(6);$$

and the labor required in the performance of the integrations indicated is simply enormous.

21. Proposed by H. W. DRAUGHON, Clinton, Louisiana.

From one corner of a square field, a boy runs in a random direction, with a random uniform velocity. The greatest distance the boy can run in one minute is equal to the diagonal of the field. What is the probability that the boy will be in the field at the end of one minute?

Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.



Let $AB=a$, and $BP=x$; then $AP=\sqrt{(x^2+a^2)}=w$, and $AC=a\sqrt{2}=m$. The boy will be in the field at the expiration of $t=1$ minute, if v be not greater than $\sqrt{(x^2+a^2)}$. Hence the required probability becomes

$$\mathbf{P}=\int_0^a\int_0^w dxdv\div\int_0^a\int_0^m dxdv,=\frac{1}{a^2\sqrt{2}}\int_0^a\sqrt{(x^2+a^2)}dx$$

$$=\frac{1}{a^2\sqrt{2}}\left[\frac{x\sqrt{(x^2+a^2)}}{2}+\frac{a^2\log[x+\sqrt{(x^2+a^2)}]}{2}\right]_0^a=\tfrac{1}{2}\left[1+\frac{\log(\sqrt{2}+1)}{\sqrt{2}}\right].$$

O. W. Anthony gets as a result $\frac{2}{\pi}$. Professor *MATZ* furnished two solutions.

22. Proposed by ALTON L. SMITH, Instructor in Drawing, Polytechnic Institute, Worcester, Massachusetts.

In a series of counts of the votes on a legislative act relative to the city of Worcester, the following results were obtained:

	YES	NO
1st count	5566	5511
2nd "	5519	5558
3rd "	5546	5517
4th "	5512	5551
5th "	5512	5541

What is the probability that the last count (the 5th) is correct?

Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor, College, New Windsor, Maryland.

I. Since the counts taken independently must be either correct or incorrect the probability that the fifth count is *incorrect* is $P'_5=\frac{1}{2}/(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2})=\frac{1}{5}$. Hence the probability that the fifth count is *correct* is $P_5=1-P_5'=\frac{4}{5}$.

II. According to *The Law of Experience*, the second count should show a *greater* probability as to correctness and a *smaller* probability as to incorrectness than the first count shows; that is, the probability as to the correctness of the fifth count should be greater than is the *similar* probability with respect to any other count *lower* than the fifth. This Law, according to the notation adopted, gives $P_1'=\frac{1}{2}, P_2'=\frac{1}{4}, \ldots, P_5'=\frac{1}{32}=(\frac{1}{2})^5$. Hence $P_5=1-(\frac{1}{2})^5=\frac{31}{32}$, which is the probability that the fifth count is correct.

PROBLEMS.

29. Proposed by JOHN DOLMAN, Jr., Philadelphia, Pennsylvania.

Neglecting perturbations, what is the average distance of the earth from the sun?

30. Proposed by F. P. MATZ, M. A., M, Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Find the average area of all the triangles that can be inscribed in a given circle.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

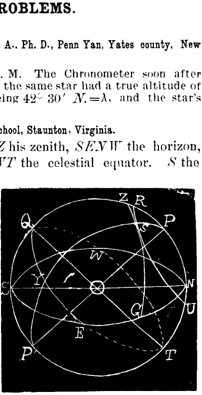
SOLUTIONS OF PROBLEMS.

17. Proposed by SAMUEL HART WRIGHT, M. D., M. A., Ph. D., Penn Yan, Yates county, New York.

A bright star passed my meridian at 7 P. M. The Chronometer soon after ran down and stopped, but I set it again when the same star had a true altitude of $30^\circ=\alpha$. What time was it then, my latitude being $42^\circ\ 30'\ N.=\lambda$, and the star's Declination $60^\circ\ N.=\delta$?

Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let O be the place of the observer, Z his zenith, $SENW$ the horizon, P and P' two poles of the heavens, $EQWT$ the celestial equator. S the star. YS the star's declination. $\angle SPZ$ the star's hour angle=arc QY, $SZPN$ the meridian. ZSG arc of star's vertical arc, SG star's altitude. From the spherical triangle ZPS we get $\cos ZS=\cos PS\cos PZ+\sin PS\sin PZ\cos ZPS$. But $ZS=90^\circ-\alpha$, $PS=90^\circ-\delta$, $PZ=90^\circ-\lambda$, $\angle ZPS=h$.



$$\therefore\ \cos h=\frac{\sin\alpha-\sin\lambda\sin\delta}{\cos\lambda\cos\delta}$$

$$=\frac{\sin 30^\circ-\sin 42\tfrac{1}{2}^\circ\sin 60^\circ}{\cos 42\tfrac{1}{2}^\circ\cos 60^\circ}$$

or we might use the formula

$$\sin^2\tfrac{1}{2}h=\tfrac{1}{2}\,\frac{\cos(\lambda-\delta)-\sin\alpha}{\cos\lambda\cos\delta}$$

$$=\tfrac{1}{2}\frac{\cos 17\tfrac{1}{2}^\circ-\sin 30^\circ}{\cos 42\tfrac{1}{2}^\circ\cos 60^\circ}.\quad h=103^\circ\ 20'\ 37''.93$$

$$=6\text{ hr. }53\text{ m. }22.53\text{ sec.}$$

$\therefore$ time=53 m. 22.53 sec. after 1 o'clock A. M.

In the figure the star is east of the meridian while in its true position it is west of the meridian.

Also solved by *Professors MATZ, SCHEFFER, WHITAKER and the PROPOSER.*

18. Proposed by M. C. STEVENS, A. M., Professor of Mathematics, Purdue University, Lafayette, Indiana.

"Show generally that a system of confocal conics is self-orthogonal."—*Johnson's Differential Equations.*

Solution by WILLIAM WOOLSEY JOHNSON, M. A., Member of the London Mathematical Society, and of the American Mathematical Society, Professor of Mathematics in the United States Naval Academy, Annapolis, Maryland.

The equation of a system of confocal conics, foci at $(\pm c, 0)$, is $\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1\ldots(1)$, a being the arbitrary constant. Differentiating, and putting p for dy/dx, $\frac{x}{a^2}+\frac{yp}{a^2-c^2}=0$, or $a^2(x+yp)=c^2x\ldots(2)$. Eliminating a between (1) and (2), $\frac{x(x+yp)}{c^2}-\frac{y(x+yp)}{c^2p}=1$, or $xyp^2+(x^2-y^2-c^2)p-xy=0$ $\ldots(3)$, which is the differential equation of the system. Putting $p=-\frac{1}{p'}$, we have $xyp'^2+(x^2-y^2-c^2)p'-xy=0\ldots(4)$, which is the equation of the trajectory when $p'=dy/dx$, and it is identical with equation (3). In other words, the roots of equation (3), as a quadratic for p, are negative reciprocals.

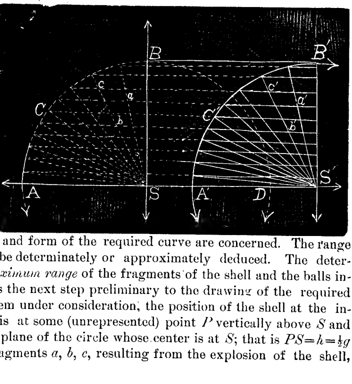
Also solved by *Professors MATZ and ZERR.*

19. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

A spherical shrapnel-shell is moving in a horizontal direction, with a constant velocity $V_1=1500$ feet per second. The shell explodes at a height $h=\frac{1}{2}g$; and the fragments of the shell and the balls inclosed by the shell, are *equably* scattered with a uniform velocity $V_2=1200$ feet per second. Draw the curve bounding *the minimum surface on the earth* on which the fragments of the shell and the balls inclosed by the shell, have fallen.

Solution by the PROPOSER.

The path of the shell is an inverted catenary of equal (or uniform) strength. At the point of maximum altitude of the path, $h=\frac{1}{2}g$, the shell moves in a horizontal direction. Ignore all such *complicating factors* as the resistance of the atmosphere, the rotary motion and geodesic contour of the earth, etc., as practically nugatory in



so far as the position and form of the required curve are concerned. The range of the shell can now be determinately or approximately deduced. The determination of the *maximum range* of the fragments of the shell and the balls inclosed by the shell, is the next step preliminary to the drawing of the required curve. In the problem under consideration, the position of the shell at the instant of explosion is at some (unrepresented) point P vertically above S and perpendicular to the plane of the circle whose center is at S; that is $PS=h=\frac{1}{2}g$ $=16\frac{1}{12}$ feet. The fragments a, b, c, resulting from the explosion of the shell,

fall on the earth at a', b', c'. The required curve, therefore, is a circle, as it should be; and the radius of this circle is the maximum range of the *fragments* and *balls*. Of course, if the atmospheric resistance and other complicating factors be not ignored, it is reasonable that the required curve should be somewhat *oval-shaped*—thus, to a certain extent, resembling the apparent disc of the rising sun or of the rising full moon.

20. **Proposed by SAMUEL HART WRIGHT, M. D., M. A., Ph. D., Penn Yan, Yates County, New York.**

When does the Dog-Star and the Sun *rise together* in latitude $\lambda=+42^\circ\ 30'$, if the Right Ascension of the said star be $\alpha=6$ hrs., 40 min., 30 sec., and the declination $\delta=-16^\circ\ 33'\ 56''$?

Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

In order that "The Dog Star" and the Sun may rise together, their *hour-angles at the instant of rising* must be the same. According to *Chauvenet's Spherical and Practical Astronomy*, Vol. I., p. 218, Art. 153, we have the formula:

$$t=\pm[-\tan\lambda\tan\delta],=74^\circ\ 10'\ 57''.77,=4h.\ 56m.\ 43.851\ sec.$$

Therefore, the next two *critical dates*; that is, the next two dates on which the *cosmical rising* of "The Dog Star" is possible in latitude $\lambda=+42^\circ\ 30'$, must be May 2, 1895, and August 2, 1895. By the Right Ascension of "The Dog-Star," as given in the problem, we are led to consider August 2, 1895, as the required date.

NOTE—The next two critical dates with respect to the *cosmical setting* of "The Dog-Star," evidently, are May 9, 1895, and August 11, 1895; and of these dates, the required one is May 9, 1895.

Also solved by *Professor ZERR* and the *PROPOSER*.

NOTE—No one of our contributors has as yet been able to effect a full and satisfactory solution to problem 21.

OUTLINE OF INVESTIGATION FOR ASYMPTOTES.

(Continued from page 185.)

In $Y=y-x\dfrac{dy}{dx}$, sub. value of $\dfrac{dy}{dx}$, gives

$$Y=\frac{4ay^2-5xy^2-3x^3}{4ay-2xy}=\frac{4a-5x-\dfrac{3x^3}{y}}{\dfrac{4a}{y}-\dfrac{2x}{y}}=\frac{4a-5x}{0}=\frac{q}{0}=\infty,$$

since $y\doteq\infty$ at limit.

Then by(c) under B, sub. values of X and Y, in $\dfrac{x}{a}+\dfrac{y}{b}=1$, gives

$$\frac{x}{2a}+\frac{y}{\infty}=1.\quad\therefore\ \frac{x}{2a}+0=1.\quad\therefore\ x=2a \text{ which is the equation sought.}$$

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

Papers for the Mathematical Congress at Kazan.

On the occasion of the dedication of the Lobachevski monument at Kazan will be held a mathematical congress of a week's duration.

It is very much desired by the managment that some papers may be contributed by Americans.

As a complete program of the scientific communications to be made in the sessions will be issued this coming February, it is not too early to solicit American scientists to think of preparing something for this memorable occasion.

Dr. George Bruce Halsted has been asked by President Vasiliev to act for him in this matter, to correspond on questions of detail with any who hope to attend the Congress in person, to take charge of the communications of those who do not anticipate being present and to guarantee their proper presentation.

Note on the May Number of the American Mathematical Monthly.

By GEORGE BRUCE HALSTED.

Mr. Warren Holden cites Halsted's Lobachevski's Geometry, but evidently has not read it, since he makes the common *petitio principii* of assuming that the equidistantial is a straight line. Mr. J. N. Lyle is as usual hopelessly muddled. Every one else knows that Beltrami proved Lobachevski's triangle and two-dimensional geometry can exist not only in pseudo-spherical space but also in Euclidean space. To apply to the immortal Helmholtz the sentence "This performance is plainly pseudo-logical," is simply sickening.

EDITORIALS.

Between July 1st and August 10th address all communications to B. F. Finkel, Chicago University.

We are pained to note the death of Dr. Daniel Kirkwood which took place at Los Angles, California, June 11. For a brief sketch of Dr. Kirkwood see the May No., Vol. I.

Editor Finkel has been elected Professor of Mathematics and

Physics in Drury College, one of the best institutions of learning in the State of Missouri. During the summer, Professor Finkel will attend the Chicago University at which institution he has been assigned a graduate Scholarship.

ON JUNE 6, from New Windsor College, Professor Hudson A. Wood, of Stevens Institute of Technology, New Jersey, because of great mathematical knowledge, success in mathematical teaching, invention of a *Perpetual Calendar* (published by A. S. Barnes & Co., New York) and manuscript of an extensive treatise on Plane and Spherical Trigonometry—book soon to be published; and Professor G. B. M. Zerr because of his high mathematical attainments as evinced by numerous contributions to the MONTHLY and on account of a thesis on "The Centroid of Surfaces," received *Cum summa laude* the degree of *Doctor of Philosophy.*

WE ARE pleased to note that our valued contributor, Professor G. B. M. Zerr, has been elected Vice-President of the Inter State College, Texarkana, Texas. He will also fill the chair of Mathematics and Sciences in that institution. Professor Zerr has lately received the degree of Ph. D. from a reputable College.

PROFESSOR W. W. LANDIS, of Thiel College, Greenville, Pennsylvania, has just been elected Professor of Mathematics in Dickinson College, Carlisle, Pennsylvania. Professor Landis always takes a lively interest in the MONTHLY.

THERE will be no MONTHLY issued during the month of July. A double number, July-August No., however, will be issued in August.

BOOKS AND PERIODICALS.

Elements of Plane and Solid Geometry: By John Macnie, A. M., Author of "Theory of Equations." Edited by Emerson E. White, A. M., LL. D., Author of *White's Series of Mathematics.* 8vo. cloth and leather back. 374 pp. Price, $1.25. Chicago and New York: American Book Company.

This text-book on geometry is worthy a place among the very best works published in America. We are sorry to note that this late work contains an erroneous demonstration of the proposition: *Triangular pyramids having equivalent bases and equal altitudes are equivalent.* This erroneous demonstration should not consign the book to oblivion as there are but very few texts on geometry that have a correct demonstration. For a discussion of the fallacy in the demonstration see p. 67 March No., Vol. I. A very commendable feature of this book is the large collection of well selected original propositions, there being 972 scattered throughout the book. We recommend it to all teachers needing a good book on geometry.

A Practical and Complete English Grammar. By J. G. Park, Instructor in English Grammar, Logic, Mental and Moral Philosophy, Ohio Nor-

mal University, Ada, Ohio. 8vo cloth. 274 pp. Price, $0.65. New York and Chicago: American Book Co.

The first 49 pages of this grammar are devoted to Language Lessons. In the remaining part of the book, Professor Park has given to the general teaching public, his admirable method of teaching grammar. Those who are personally acquainted with his method know that, for power of creating interest and enthusiasm in classes ranging from 300 to 350 students, it is unsurpassed. We believe that this grammar will be epoch-making in the teaching of grammar. As to the real technique of grammatical constructions, there may be some points on which the best philologists will differ from Prof. Park, but upon the whole his book will bear careful perusing. We trust that this book may supplant the many useless and fossiliferous texts that are preventing a thorough mastery of the English Language and creating a disgust for its study in many of the schools in this country.

The Review of Reviews. An International Monthly Magazine. Edited by Albert Shaw. Price, $2.50 per year. Single Number, 25 cents. The Review of Reviews Co.: New York City.

This Magazine is one of the best of our day. All the current events of the world are briefly noted while the more important events are thoroughly discussed. the leading articles in the leading magazines of the world are briefly reviewed. The June No. of the Review of Reviews contains nearly the whole of Dr. Halsted's biography of Prof. Cayley. This extract of the biography was taken from the April No. of the MONTHLY.

American Journal of Mathematics. Edited by THOMAS CRAIG, with the co-operation of SIMON NEWCOMB. Quarterly. 4to. $5 per volume. Johns Hopkins Press. Baltimore, Maryland.

The July number of this valuable publication contains the following papers: "On Irrational Covariants of certain Binary Forms," by E. Study; "On the Connection between Binary Quartics and Elliptic Functions,"by E. Study; "Semi-Combinants as Concomitants of Affiliants," by Henry S. White; "Simplification of Gauss's third Proof that every Algebraic Equation has a Root,"by Maxime Bocher;"Note sur lignes cycloidales," par Rene de Saussure; "Note on Lines of Curvature," by Thomas Hardy Taliaferro.

Annals of Mathematics. Ormond Stone, Editor, University of Virginia. The January number contained "Literal Expression for the Motion of the Moon's Perigee," by G. W. Hill; "Note on Gregory's Discussion of the Treatment of the Conditions for an Umbilicus," by Angelo Hall; "Concerning the Definition by a system of Functional Properties of the Function $f(z)=\dfrac{\sin \pi z}{\pi}$, by E. H. Moore. Twelve problems are solved, and seventeen exercises proposed.

Journal de Mathematiques Elementaires. Publie par H. Vuibert. Paris: Librairie Nony et Cie 17, rue des Ecoles.

The May number contains the usual amount of interesting matter, including the solutions of problems in Arithmetic, Algebra, Geometry, Trigonometry, and Physics. Several new problems are proposed.

L'Intermediaire des Mathematiciens. Dirige par C.-A. Laisant, et Emile Lemoine. Monthly. Paris.

The May number of this interesting Journal contains Questions 550 to 570, and publishes ten pages of answers to questions previously proposed.

El Progreso Matematico. Periodico de Matematicas Puras y Aplicadas. Director: Don Zoel G. de Galdeano, catedratico de la Universidad de Zaragoza.

The March number of this excellent Monthly has several interesting papers, and many fine problems and solutions.

Periodico di Matematica per L'Insegnamento Secondario. Publicato per cura di Aurelio Lugli, Professore di Matematica nel R. Instituto tecnico di Roma. Via Panisperna, 69-Roma.

In the January-February number several important papers are published, and quite a number of questions are proposed and many others solved.

The Educational Times, London, for June has the usual amount of space devoted to its Mathematical Department. Fifteen solutions appear, and 37 new problems are proposed, several of which are republished from the MONTHLY.

The Kansas University Quarterly for April has been received. Nine very interesting articles are published, but no mathematical contributions appear in this number.

The Monist. The April number contains several important papers by well-known writers. Nearly every issue of this valuable Quarterly has one or more contributions of a mathematical character.

Miscellaneous Notes and Queries. A Monthly Magazine of History, Folk-Lore, Mathematics, Mysticism, Art, Science, etc.

The June Issue contains much varied and interesting information gathered from many sources.

NOTE:—In the cut used in the fifth solution on page 190, by mistake the letter P was omitted at a point on the upper part of the line AD. Readers of the MONTHLY will please note this error.